Example:

$$dx(t) = \mu\, dt + \sigma x(t)\, dw(t)$$

$$y(t) = e^{x(t)}$$

$dy(t)$

i) starting process: $x(t)$

$\mu = \mu$; $\sigma = \sigma x(t)$

$f(x) = e^{x}$

ii) starting process: $w(t)$

$\mu = 0$; $\sigma = 1$

$f(w) = ??$

$\mathrm{Cov}(w(t), w(s)) = s$

$s < t$

- $\mathrm{Cov}(w(t), w(s)) = \mathrm{Cov}(w(t) - w(s) + w(s), w(s))$

$= \mathrm{Cov}(w(t) - w(s), w(s)) + \mathrm{Cov}(w(s), w(s))$

$= 0 + \mathrm{Var}(w(s)) = s$

- $\mathrm{Cov}(w(t), w(s)) = E[w(t) w(s)]$

$X \perp\!\!\!\perp Y \Rightarrow \mathrm{Cov}(X, Y) = 0$

$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0$

independence $\Leftrightarrow f_{X,Y}(x, y) = f_X(x) f_Y(y)$, $\forall x, y$

$$E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_{X,Y}(x,y)\, dy\, dx$$

$$\underset{\text{indep}}{=} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_X(x)\, f_Y(y)\, dy\, dx$$

$$= \left(\int_{-\infty}^{+\infty} x f_X(x)\, dx\right)\left(\int_{-\infty}^{+\infty} y f_Y(y)\, dy\right)$$

$$= E[X]\, E[Y]$$

$\{W(t), t \geq 0\}$

$\{B(t) = W(t) + t, t \geq 0\}$

- $B(0) = W(0) + 0 = 0$
- Continuous sample-path: $\{W(t), t \geq 0\}$ has continuous sample paths + $\{t, t\geq 0\}$, which is also continuous. Thus it has continuous sample path
- non-differentiability: don't worry!
- $W(t) \sim \mathcal{N}(0, t)$
  $W(t) + t \sim \mathcal{N}(t, t)$
- independence of increments

$$\mathrm{Cov}(B(t) - B(s), B(s)) = \mathrm{cov}(W(t) + t - W(s) - s, W(s) - s)$$

$t > s$

$$= \mathrm{Cov}(W(t), W(s)) - \mathrm{Cov}(W(s), W(s))$$

$$\mathrm{Cov}(X, a) = E[aX] - E[a]E(X) = aE(X) - aE(X) = 0$$

$$= s - \mathrm{Var}(W(s)) = s - s = 0$$

$\Rightarrow$ independence of increments

- $\mathrm{Var}(B(t+s) - B(s)) =$ depends only on t

$$\mathrm{var}(B(t) - B(s)) = \text{depends only on } t-s$$

$$= \mathrm{var}(B(t)) + \mathrm{var}(B(s)) - 2\,\mathrm{cov}(B(t), B(s))$$

$$= t + s - 2\,\mathrm{cov}(w(t)+t,\ w(s)+s)$$

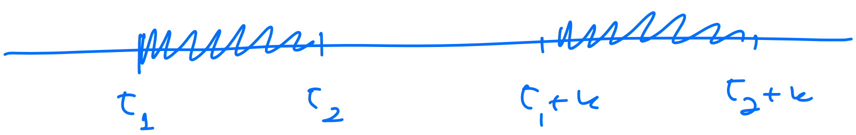
$$= t + s - 2\,\mathrm{cov}(w(t), w(s))$$

$$= t + s - 2s = t - s \quad //$$

$$\Rightarrow B(t) - B(s) \sim N(t-s,\ t-s)$$

$$B(t-s) \sim N(t-s,\ t-s)$$

$$B(t_2) - B(t_1) \overset{D}{=} B(t_2+u) - B(t_1+u)$$



$$\sim N(-,-) \qquad\qquad \sim N(-,-)$$